Final report

Introduction

This report aims to conclude the analytic way to solve the wave equation under center potential of Hydrogen Atom. We use methods of separation variables to obtain the independent equation of radial and angular part. For radial part, Zi jun has used polynomials and analyze the energy levels. And for angular part ,Ruo han yang used two methods, one is start from the angular momentum’s eigen function ,and another is start from the solution with Legendre equation.Finally, we present a visualized hydrogen atom using matlab to better understand its eigenstates.

Angular part

The spherical coordinates ()

The Laplacian takes the form:



And the Schrodinger equation reads:



With the separated methods (The equation of Y must equal a constant), and the constant could be written as ,



Multiply the equation with :



Which is equivalent to:



Use the separation again:



The equation could change to:



The Azimuthal equation is easy to solve:





To solve ,Here offer 2 methods:

**1.Angular momentum ladder’s Methods:**

To solve the polynomial ,the first method could start from Angular Operators:

Because the orbital angular momentum operator is:



and could be loosely written as:

,

**1.1 Angular momentum operator:**

In spherical coordinates it could be written as



We know the following properties:

1. the ladder operator is:





1. The commutation relation is:



1. And From easy calculation we could obtain:



So they shared a same eigen function.

1. Given the definitions of angular momentum eigenstates from:



translate to the differential equations:



The latter equation is easy to solve: the azimuth dependence of the spherical

harmonics must be

**1.2 Derivation of Spherical Harmonics:**

The angular momentum couldn’t be lower so the operator acting on the eigen functions must be zero:



On the other hand, we know already that:



This equation is solved easily, by writing it as:



with an overall normalization constant c.

Putting things together, we find



**1.3.Normalization:**

The absolute value of c can be fixed by the normalization:





, and further change the variable to 



It is could be evaluated by Beta function！



So:



The constant is:



Putting them all together:



**1.4.Use Ladder operator to Get Higher m term:**

Now we got ,next is keep acting raising operatoron it and obtain all ,

We got the relation from analyze the angular momentum operator and obtained:



So the it could be obtained that:



Act each on :





Here we have cancelled factors of and . The next trick we need is to write



Then:



And .

So this is identical with the Legendre Polynomials:



**2. Sturm–Liouville theory Methods**



it is the Associated Legendre polynomials:



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**2.1 Legendre equation:**

First, we need solve the Legendre equation.

Write the function with the **Sturm–Liouville form:**



If y could be expanded as the sum of the polynomial of x:



Substitute it to the equation:



So the relation of  and is determined:



And the y is the sum of and ,the odd terms and even terms:



Then the convergence of Legendre function is 1,

So if it need to be bounded, it must vanish after a few terms:

Given a term such as

，

The infinite series will be cut off if , So, 

But the if l is even, the other term of odds are infinite series, so The General solution of Legendre equation is the linear combination of one [polynomial](javascript:;)  and the other .



If the Highest term  can be determined, then all the other term could be determined as well:

If ,



[legendre polynomial](javascript:;) could be the most simplest:



So:



The first few terms of Polynomial is:



**2.2. Associated Legendre polynomials**

Finally, We can use the Legendre equation to solve this Angular equation .



Differentiate it once, twice, and th:





And it also equals to:



If we change the variable



Substitute this to the equation, we could obtain:



Which is the solution of 

in conclusion, The solution of Angular equation is

